On the Invariant Properties of Hyperbolic Bivariate Third-Order Linear Partial Differential Operators

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Abstract. Bivariate, hyperbolic third-order linear partial differential operators under the gauge transformations $L \to g(x,y)^{-1} \circ L \circ g(x,y)$ are considered. The existence of a factorization, the existence of a factorization that extends a given factorization of the symbol of the operator are expressed in terms of the invariants of some known generating set of invariants. The operation of taking the formal adjoint can be also defined for equivalent classes of LPDOs, and explicit formulae defining this operation in the space invariants were obtained.

1 Introduction

Nowadays, constructive factorization algorithms are greatly in demand, being used in recent algorithms for the exact solution of Linear Partial Differential Equations (LPDEs). For example, they are used in the numerous generalizations and modifications of the 18th-century Laplace-Transformations Method, in the Loewy decomposition method, and in other methods (see for example [1–6]). Both the property of having a factorization, and the property of having a factorization that extends a certain factorization of the (principal) symbol are invariant under Gauge transformations of LPDOs, *viz.* $L \to g(x,y)^{-1} \circ L \circ g(x,y)$, and therefore can be described invariantly in terms of the invariants of a generating set of invariants, if such a set is known.

The Laplace Transformations Method [7] is an example of the use of an invariant description of factorization properties for a second-order hyperbolic operator. The normalized form of such operators is

$$L = D_x \circ D_y + aD_x + bD_y + c \ , \tag{1}$$

where all the coefficients are functions of x and y, and the *Laplace invariants*

$$h = c - a_x - ab, \quad k = c - b_y - ab \tag{2}$$

form a generating set of invariants with respect to the Gauge transformations. It is easy to see that L is factorable if and only if h or k is zero. Moreover, the factorization of the principal symbol $\mathrm{Sym}(L) = X \cdot Y$ can be extended if and only if $h = 0$, while $\mathrm{Sym}(L) = Y \cdot X$ can be extended if and only if $k = 0$.

The method of Laplace starts with an initial operator L and applies two transformations $L \to L_1$ and $L \to L_{-1}$ called *Laplace transformations* until one of the transformed operators is factorable (the Laplace transformations are admitted by operators of the form (1)). The Laplace invariants of the transformed operators L_1 and L_{-1} can be expressed in terms of the invariants of the initial operator:

$$h_1 = 2h - k - \partial_{xy}(\ln |h|),\ k_1 = h,\ h_{-1} = k,\ k_{-1} = 2k - h - \partial_{xy}(\ln |k|)\ .$$

So assuming that L is not factorable, and so $h \neq 0, k \neq 0$, only one invariant for each of the transformed operators can vanish. In such the way, instead of a sequence of operators, one considers the chain of their Laplace invariants

$$\ldots \leftrightarrow k_{-2} \leftrightarrow k_{-1} \leftrightarrow k \leftrightarrow h \leftrightarrow h_1 \leftrightarrow h_2 \leftrightarrow \ldots\ . \tag{3}$$

One iterates the Laplace transformations until one of the Laplace invariants in the sequence (3) vanishes. In this case, one can solve the corresponding transformed equation in quadratures and then use the inverse substitution to obtain the complete solution of the original equation. What is more, one may prove (see for example [8]) that if the chain (3) is finite in both directions, then one may obtain a quadrature-free expression for the general solution of the original equation.

In the case considered by Laplace, the invariants h and k can be simply obtained from the incomplete factorizations, $L = (D_x + b) \circ (D_y + a) + h = (D_y + a) \circ (D_x + b) + k$. That is why the invariant necessary and sufficient conditions of factorizations becomes so simple ($h = 0$ or $k = 0$). For hyperbolic operators of the next order — order three — the situation become much more difficult: the "remainder" of an incomplete factorization is not invariant in the generic case, and the invariant conditions are not trivial.

In the present paper we find invariant necessary and sufficient conditions of factorizations extending given (we consider all the possibilities) factorizations of the principal symbol of third-order bivariate hyperbolic linear partial differential operators. These invariant conditions are given in terms of invariants of the generating set of invariants found in [9]. Also in the scope of the paper we investigate the classical operation of taking the formal adjoint of an operator, define it on the equivalent classes of the considered LPDOs, and obtain explicit formulae in the space of invariants. Some instances of the latter result allow us to reduce the number of case considerations when finding an invariant definition of the property of the existence of a factorization.

The paper is organized as follows. In Section 2 preliminaries facts and definitions are given. In Section 3 we discuss connections between factorization of LPDOs and invariants of a family of LPDOs under the gauge transformations, also we show how we reduce the number of factorization types to consider to just four ones. In Sections 4, 5, and 6, the existence of factorizations of these four factorization types has been expressed in terms of invariants of the generating system of invariants found in [9]. In Section 7 the operation of taking the formal adjoint is defined in the space of invariants.

2 Definitions and Notations

Consider a field K with commuting derivations ∂_x, ∂_y acting on it. Consider the ring of linear differential operators $K[D] = K[D_x, D_y]$, where D_x, D_y correspond to the derivations ∂_x, ∂_y, respectively. In $K[D]$ the variables D_x, D_y commute with each other, but not with elements of K. For $a \in K$ we have the relation $D_i a = a D_i + \partial_i(a)$. Any operator $L \in K[D]$ is of the form $L = \sum_{i+j=0}^{d} a_{ij} D_x^i D_y^j$, where $a_{ij} \in K$. The polynomial $\mathrm{Sym}_L = \sum_{i+j=d} a_{ij} X^i Y^j$ in formal variables X, Y is called the (principal) *symbol* of L. An operator $L \in K[D]$ is said to be *hyperbolic* if its symbol is completely factorable (all factors are of first order) and each factor has multiplicity one.

Let K^* denote the set of invertible elements in K. For $L \in K[D]$ and every $g \in K^*$ consider the gauge transformation $L \to g^{-1} \circ L \circ g$. Then an algebraic differential expression I in coefficients of L is *invariant* under the gauge transformations (we consider only these in the present paper) if it is unaltered by these transformations. Trivial examples of invariants are the coefficients of the symbol of the operator. A generating set of invariants is a basis in which all possible differential invariants can be expressed.

We use the usual abbreviations: LPDO for Linear Partial Differential Operator, LPDE for Linear Partial Differential Equation.

3 Factorization via Invariants

Any hyperbolic third-order LPDO in some system of coordinates has the form

$$L = (pD_x + qD_y)D_x D_y + \sum_{i+j=0}^{2} a_{ij} D_x^i D_y^j \ , \tag{4}$$

where all the coefficients belong to K (they are some functions of x and y) and where $p, q \neq 0$.

Remark 1. Note that the normalized form of such operators is slightly simpler than above, namely, one can put without loss of generality $p = 1$. The introduction of the parameter p makes all the reasoning symmetric with respect to x and y, and therefore reduces the number of cases requiring consideration on the way to our main goal.

Operators of the form (4) admit gauge transformations, and p, q are the trivial invariants.

Theorem 1. *[9] The following form a generating set of invariants for operators of the form (4):*

$$
\left.\begin{array}{l}
I_p = p \ , \\
I_q = q \ , \\
I_1 = 2q^2 a_{20} - q a_{11} p + 2 a_{02} p^2 \ , \\
I_2 = -q p^2 a_{02y} + a_{02} p^2 q_y + q^2 a_{20x} p - q^2 a_{20} p_x \ , \\
I_3 = a_{10} p^2 + (2 q_y p - 3 q p_y) a_{20} + a_{20}^2 q - a_{11y} p^2 + a_{11} p_y p + q p a_{20y} \\
\qquad -a_{11} a_{20} p \ , \\
I_4 = a_{01} q^2 + (2 q p_x - 3 p q_x) a_{02} + a_{02}^2 p - a_{11x} q^2 + a_{11} q q_x + q p a_{02x} \\
\qquad -a_{02} a_{11} q \ , \\
I_5 = a_{00} p^3 q + 2 a_{02} p^3 a_{20x} - 2 q^2 a_{20}^2 p_x - a_{02} a_{10} p^3 - a_{01} a_{20} p^2 q \\
\qquad + \frac{1}{2} a_{11x} p_y p^2 q + \frac{1}{2} a_{11y} p_x p^2 q + (\frac{1}{2} p_{xy} p^2 q - p_x p_y p q) a_{11} \\
\qquad + a_{11} p q a_{20} p_x - \frac{1}{2} a_{11xy} p^3 q + (q q_x p^2 - q^2 p_x p) a_{20y} - 2 a_{02} p^2 a_{20} p_x \\
\qquad - a_{11} p^2 q a_{20x} + (q p^2 q_y - p q^2 p_y) a_{20x} + 2 q^2 a_{20} a_{20x} p + \\
\qquad (q q_{xy} p^2 - q^2 p_{xy} p + 4 q^2 p_x p_y - 2 q p_x q_y p - 2 q q_x p p_y) a_{20} \\
\qquad + a_{20} a_{11} a_{02} p^2 \ .
\end{array}\right\} \qquad (5)
$$

Any set of values of these invariants uniquely defines an equivalent class of operators of the form (4). All the invariant properties of such operators can be described in terms of the invariants of the above generating set.

Lemma 1. *The property of having a factorization (or a factorization extending a certain factorization of the symbol) is invariant.*

Proof. Let $L = F_1 \circ F_2 \circ \ldots \circ F_k$, for some operators $F_i \in K[D]$. For every $g \in K^*$

$$g^{-1} \circ L \circ g = \left(g^{-1} \circ F_1 \circ g\right) \circ \left(g^{-1} \circ F_2 \circ g\right) \circ \ldots \circ \left(g^{-1} \circ F_k \circ g\right) \ ,$$

and since the gauge transformations do not alter the symbol of an LPDO, we prove the statement of the theorem.

Remark 2. Recall that as for two LPDOs $L_1, L_2 \in K[D]$ we have

$$\mathrm{Sym}_{L_1 \circ L_2} = \mathrm{Sym}_{L_1} \cdot \mathrm{Sym}_{L_2} \ ,$$

any factorization of an LPDO extends some factorization of its symbol. In general, if $L \in K[D]$ and $\mathrm{Sym}_L = S_1 \cdot \ldots \cdot S_k$, then we say that the factorization

$$L = F_1 \circ \ldots \circ F_k, \qquad \mathrm{Sym}_{F_i} = S_i, \ \forall i \in \{1, \ldots, k\},$$

is of the *factorization type* $(S_1) \ldots (S_k)$.

Consider all possible factorizations of the symbol of an LPDO (4), namely $\mathrm{Sym}_L = (pX + qY)XY$. Owing to the non-commutativity of LPDOs one has to consider factorizations of the polynomial Sym_L assuming that factors do not commute. Thus $\mathrm{Sym}_L = (pX + qY)XY$ has 12 different factorizations:

$$(S)(XY) \ ,$$

$$
\begin{aligned}
&(XY)(S) \ ,\\
&(X)(YS) \ , \ (Y)(XS) \ ,\\
&(YS)(X) \ , \ (XS)(Y) \ ,\\
&(S)(X)(Y) \ , \ (S)(Y)(X) \ ,\\
&(X)(S)(Y) \ , \ (Y)(S)(X) \ ,\\
&(X)(Y)(S) \ , \ (Y)(X)(S) \ ,
\end{aligned}
$$

where $S = (pX + qY)$. By Remark (1) it is enough to consider one of the factorizations for each of the lines of the list above. Thus, there are seven cases to consider. Proceeding further, we can almost half this number of cases (i.e. 7 cases) once we know how to express generating invariants of the formal adjoint $L^\dagger$ of an LPDO L in terms of generating invariants of L. In Section 7 we find such formulae, and so only the the following cases need to be considered:

$$
\begin{aligned}
&(S)(XY) \ ,\\
&(X)(YS) \ ,\\
&(S)(X)(Y) \ ,\\
&(X)(S)(Y) \ .
\end{aligned}
$$

4 Factorization Type $(pX + qY)(XY)$

Theorem 2. *Consider an equivalent class of (4) given by the values of the invariants I_1, I_2, I_3, I_4, I_5 (5). The operators of the class have a factorization of the factorization type $(pX+qY)(XY)$ if and only if the following two conditions hold.*

$$
\begin{aligned}
&I_3q^3 - I_4p^3 + pq(pI_{1x} - qI_{1y}) + pq(q_y - p_x)I_1 + 2(p_yq^2 - q_xp^2)I_1 - 3pqI_2 = 0 \ ,\\
&I_sI_2 + I_r + 2pq^2I_{2x} + q^3I_{2y} = 0 \ .
\end{aligned}
$$

Proof. First, using the formulae of the invariants (5), we express the coefficients a_{11}, a_{10}, a_{01}, a_{00} of (4) in terms of these invariants and a_{20}, a_{02}. We have, for example, $a_{11} = (-I_1 + 2q^2a_{20} + 2a_{02}p^2)/(pq)$, and other expressions are too large to give them here explicitly. Then an operator (4) of the class has factorization $F_{(pX+qY)(XY)} = (pD_x + qD_y + r) \circ (D_{xy} + aD_x + bD_y + c)$, where all the coefficients are functions of x and y, takes place if and only if $L - F_{(pX+qY)(XY)} = 0$. Equating the coefficients at D_{xx}, D_{xy}, D_{yy}, D_y on the both sides of this equality, one computes

$$
\begin{aligned}
&a = a_{20}/p \ , \quad b = a_{02}/q \ , \quad r = -\frac{1}{pq}\, I_1 + \frac{q^2a_{20} + a_{02}p^2}{pq} \ ,\\
&c = (I_4p^2 - qpI_{1x} + 2q^3pa_{20x} + (2q_xp + qp_x)I_1 - 2q^3p_xa_{20}\\
&+a_{02}a_{20}q^2p - q^2p^2a_{02y} + qp^2a_{02}q_y)/(q^3p^2)
\end{aligned}
$$

as p and q are known to be different from zero. While equating the coefficients of D_x and the "free" coefficients of both sides of that, we get two conditions for the existence of a factorization, which still involve the coefficients a_{20} and a_{02} and, therefore, are not invariant. On the other hand, by Lemma 1, there should be a way to describe existence of a factorization (a factorization extending certain factorization of the symbol) invariantly.

Consider the first condition, which after multiplication by p^2q^3, can be noticed to be equivalent to the following constrain for invariants of L:

$$C_{10} = I_3q^3 - I_4p^3 + pq(pI_{1x} - qI_{1y}) + pq(q_y - p_x)I_1 + 2(p_yq^2 - q_xp^2)I_1 - 3pqI_2 = 0 \; . \tag{6}$$

Consider the second condition multiplied for convenience on both sides by p^2q^4 (denote the result as $C_{00} = 0$). It is a large expression. Consider all the terms of C_{00} with second-order derivatives of a_{20}, a_{02}:

$$-2p^2q^4a_{20xx} \; , -pq^5a_{20xy} \; , 2q^3p^3a_{02xy} \; , 2p^2q^4a_{02yy} \; .$$

Thus, subtracting $2pq^2I_{2x} + q^3I_{2y}$ from C_{00}, we cancel the terms with second-order derivatives of a_{20}, a_{02}. Denote the result of the subtraction by C_{001}. Consider terms of C_{001} containing first-order derivatives of a_{20}, a_{02}:

$$q^3(I_1 + 2q^2p_y + 2qpq_y + 4p^2q_x + 4pqp_x - 3a_{02}p^2)a_{20x} \; , \tag{7}$$
$$-q^2p(I_1 + 2q^2p_y + 2qpq_y + 4p^2q_x + 4pqp_x - 3a_{02}p^2)a_{02y} \; , \tag{8}$$

and compare them with those in I_2. One can see that the ratio of the coefficient at a_{20x} in (7) to that in I_2 equals to the ratio of the coefficient at a_{02y} in (8) to that in I_2, and this ratio is

$$s = I_s - 3pqa_{02} \; ,$$

where $I_s = \frac{q}{p}(4p(qp_x+pq_x)+2q(pq_y+qp_y)+I_1)$, that is an invariant. Subtracting sI_2 from C_{001} (denote the result of the subtraction by C_{002}), we cancel all the terms contaning first-order derivatives of a_{20}, a_{02}, and get

$$C_{002} = (I_3q^3 - I_4p^3 + qp^2I_{1x} - pq^2I_{1y} + pq(q_y - p_x)I_1 + 2(p_yq^2 - q_xp^2)I_1)a_{02} + I_r \; , \tag{9}$$

where $I_r = \frac{q^3p}{2}I_{1xy} - qp^2(qI_{4y} - pI_{4x}) + \frac{q^3}{p}I_5 + q^2p^2I_{1xx} - \frac{3q^2pq_x}{2}I_{1y} + pI_1I_4 + \Big(-2qp^2q_{xx} + 6q_x^2p^2 + q^2q_xp_y + 4qpq_xp_x - q^2pp_{xx} + q^2p_xq_y - \frac{3q^2pq_{xy}}{2} + 5qpq_xq_y + 2p_x^2q^2 - \frac{q^3p_xp_y}{p}\Big)I_1 + 3p^2(qq_y + pq_x)I_4 + \Big(2q_x + \frac{qp_x}{p}\Big)I_1^2 - pq\Big(\frac{3qq_y}{2} + 2qp_x + 4pq_x\Big)I_{1x} - qI_1I_{1x}$ is an invariant. Comparing (9) with (6), one can notice that the coefficient at a_{02} in C_{002} equals $(C_{10} + 3pqI_2)$. As $C_{10} = 0$ is a necessary condition for L to be factorable with the considered factorization type, the coefficient at a_{02} in C_{002} becomes just $3pqI_2$. Which is fortunately is canceled in expression for C_{00}, when we combine the results:

$$\begin{aligned} C_{00} &= (C_{10} + 3pqI_2)a_{02} + (I_s - 3pqa_{02})I_2 + I_r + 2pq^2I_{2x} + q^3I_{2y} \\ &= C_{10}a_{02} + I_sI_2 + I_r + 2pq^2I_{2x} + q^3I_{2y} \; . \end{aligned}$$

Corollary 1 (case $p = 1$). *Consider equivalent classes of (4) possessing the property $p = 1$, and given by the values of the invariants I_1, I_2, I_3, I_4, I_5 (5). The operators of the class have a factorization of the factorization type $(X+qY)(XY)$ if and only if*

$$\begin{cases} I_3 q^3 - I_4 + q(I_{1x} - qI_{1y}) + qq_y I_1 - 2q_x I_1 - 3qI_2 = 0 \ , \\ I_s I_2 + I_r + 2q^2 I_{2x} + q^3 I_{2y} \qquad\qquad\qquad\quad = 0 \ . \end{cases}$$

where $I_s = q(4q_x + 2qq_y + I_1)$ *and* $I_r = \frac{q^3}{2} I_{1xy} - q(qI_{4y} - I_{4x}) + q^3 I_5 + q^2 I_{1xx} - \frac{3q^2 q_x}{2} I_{1y} + I_1 I_4 + \Big(-2qq_{xx} + 6q_x^2 - \frac{3q^2 q_{xy}}{2} + 5qq_x q_y \Big) I_1 + 3(qq_y + q_x) I_4 + 2q_x I_1^2 - q\Big(\frac{3qq_y}{2} + 4q_x \Big) I_{1x} - qI_1 I_{1x}$.

5 Factorization Type $(X)(YS)$

Theorem 3. *Consider an equivalent class of (4) given by the values of the invariants I_1, I_2, I_3, I_4, I_5 (5). The operators of the class have a factorization of the factorization type $(X)(pXY + qY^2)$ if and only if*

$$\begin{cases} I_4 - 2q_x p_x q + 2q_x^2 p - qpq_{xx} + q^2 p_{xx} = 0 \ , \\ -4p^2 q_x I_2 + p^2 q I_{2x} + I_r \qquad\qquad = 0 \ , \end{cases}$$

where $I_r = -3/2 q_x q p^2 I_{1y} + I_5 q^2 + \frac{1}{2} I_{1xy} q^2 p^2 - q^3 p I_{3x} + (q^2 p q_x + 2p_x q^3) I_3 + (-p_{xy} q^2 p + 3q_x p_y qp + 2q_x q_y p^2 - \frac{1}{2} q_{xy} qp^2 + p_x p_y q^2) I_1 + (-p_y q^2 p - \frac{1}{2} q_y qp^2) I_{1x}$.

Proof. The case we consider here is much easier than that of section 4. As we do there first we express $a_{00}, a_{10}, a_{01}, a_{11}$ in terms of a_{20}, a_{02} and the invariants (5). Then for an operator L (4) of the class consider a factorization of the form

$$L = (D_x + r) \circ (pD_{xy} + qD_{yy} aD_x + bD_y + c) \ , \qquad (10)$$

where all the coefficients belong to K (some functions of x and y). Substituting just found expressions for a_{00}, a_{10}, a_{01}, a_{11}, and equating the coefficients at D_{yy}, D_{xx}, D_{xy}, D_x on the both sides of (10), one computes $r = (a_{02} - q_x)/q$, $a = a_{20}$, $b = -(I_1 - 2q^2 a_{20} - a_{02} p^2 - p^2 q_x + p_x qp)/q/p$, $c = -(-I_3 q^2 + a_{20} q I_1 - a_{20}^2 q^3 - a_{20} q a_{02} p^2 + q^3 p_y a_{20} + qpI_{1y} - q^3 p a_{20y} - 2qp^3 a_{02y} - q_y p I_1 + 2q_y p^3 a_{02} - 2qp_y I_1 - qa_{20} p^2 q_x + a_{20x} q^2 p^2)/q^2/p^2$, as p and q are known to be different from zero. Equating the coefficients at D_y we get first constrain on invariants,

$$I_4 - 2q_x p_x q + 2q_x^2 p - qpq_{xx} + q^2 p_{xx} = 0 \ . \qquad (11)$$

Equating the "free" coefficients of the both sides of (10), we get a condition of existence of a factorization in particular in terms of a_{20} and a_{02}. To cancel denominators, multiply this condition on the both sides by $p^3 q^3$ (denote the result as $C_{00} = 0$). Consider all the terms of C_{00} with second-order derivatives of a_{20}, a_{02}:

$$p^3 q^3 a_{20xx} \ , -q^2 p^4 a_{02xy} \ .$$

Thus, subtracting p^2qI_{2x} from C_{00}, we kill all the terms with second-order derivatives of a_{20}, a_{02}. Denote the result of the subtraction by C_{001}. Consider terms of C_{001} containing first-order derivatives of a_{20}, a_{02}:

$$-4q_xp^3q^2a_{20x} \ , 4q_xqp^4a_{02y} \ ,$$

and compare them with those in I_2. One can see that subtracting $-4p^2q_xI_2$ from C_{001} we cancel all the terms containing first-order derivatives of a_{20}, a_{02}. Denote the result of this subtraction by C_{002}, then

$$C_{002} = (I_4qp^2 - 2q^2p^2q_xp_x + q^3p^2p_{xx} + 2qp^3q_x^2 - q^2p^3q_{xx})a_{20} + I_r \ , \qquad (12)$$

where $I_r = -3/2q_xqp^2I_{1y} + I_5q^2 + \frac{1}{2}I_{1xy}q^2p^2 - q^3pI_{3x} + (q^2pq_x + 2p_xq^3)I_3 + (-p_{xy}q^2p + 3q_xp_yqp + 2q_xq_yp^2 - \frac{1}{2}q_{xy}qp^2 + p_xp_yq^2)I_1 + (-p_yq^2p - \frac{1}{2}q_yqp^2)I_{1x}$ is an invariant. The constrain (11) implies that the coefficients at a_{02} in C_{002} is zero provided the factorization (10) takes place. Thus, combining the results, we have

$$C_{00} = -4p^2q_xI_2 + p^2qI_{2x} + I_r \ .$$

Corollary 2 (case $p = 1$). *Consider equivalent classes of (4) possessing the property $p = 1$, and given by the values of the invariants I_1, I_2, I_3, I_4, I_5 (5). The operators of the class have a factorization of the factorization type $(X)(XY + qY^2)$ if and only if*

$$\begin{cases} I_4 + 2q_x^2 - qq_{xx} & = 0 \ , \\ I_5q^2 - 4p^2q_xI_2 + p^2qI_{2x} + \frac{1}{2}I_{1xy}q^2 - I_{3x}q^3 - & \\ \frac{3}{2}q_xI_{1y}q - \frac{1}{2}q_yI_{1x}q + q_xI_3q^2 + (-\frac{1}{2}q_{xy}q + 2q_xq_y)I_1 & = 0 \ . \end{cases}$$

6 Factorization Types $(pX + qY)(X)(Y)$ and $(X)(pX + qY)(Y)$

Here we omit all the proofs as they employ similar to the section 4 ideas and are much simpler.

Theorem 4. *Consider an equivalent class of (4) given by the values of the invariants I_1, I_2, I_3, I_4, I_5 (5). The operators of the class have a factorization of the factorization type $(pX + qY)(X)(Y)$ if and only if*

$$\begin{cases} I_3q^2 - qpI_{1y} + (q_yp + 2qp_y)I_1 & = 0 \ , \\ I_4p^2 - I_1xqp + (2qxp + pxq)I_1 & = 0 \ , \\ I_5q^2 + (p_xpq^2 + \frac{1}{2}q_xp^2q)I_{1y} - \frac{1}{2}I_{1xy}p^2q^2 + (p_ypq^2 + \frac{1}{2}q_yp^2q)I_{1x} + & \\ (-3p_xp_yq^2 - p_xq_ypq + p_{xy}pq^2 + \frac{1}{2}q_{xy}p^2q - q_xp_ypq - q_xq_yp^2)I_1 & = 0 \ . \end{cases}$$

Theorem 5. *Consider an equivalent class of (4) given by the values of the invariants I_1, I_2, I_3, I_4, I_5 (5). The operators of the class have a factorization of* *the factorization type* $(X)(pX+qY)(Y)$ *if and only if*

$$\begin{cases} I_3q^2 - qpI_{1y} + q_y pI_1 + 2qp_y I_1 & = 0 \ , \\ 2pq_x^2 - qpq_{xx} + q^2 p_{xx} + I_4 - 2q_x p_x q & = 0 \ , \\ I_5 q^2 - \frac{1}{2}p^2q^2 I_{1xy} + \frac{1}{2}q_x qp^2 I_{1y} + p_x q^2 pI_{1y} + \\ (p_y q^2 p + \frac{1}{2}q_y qp^2)I_{1x} + (-q_x q_y p^2 + \frac{1}{2}q_{xy}qp^2 - \\ 3p_x p_y q^2 + p_{xy}q^2 p - q_x p_y qp - p_x q_y qp)I_1 & = 0 \ . \end{cases}$$

7 Formal Adjoint

In this section we consider the operation of taking the formal adjoint of an LPDO, and define such operation on the equivalent classes of third-order bivariate non-hyperbolic LPDO. At the end of the section we apply this knowledge to complete the cases' consideration in the finding of invariant condition of the property of the existence of a factorization of certain factorization type.

For an operator $L = \sum_{|J| \leq d} a_J D^J$, where $a_J \in K,\ J \in \mathbf{N}^n$ and $|J|$ is the sum of the components of J, the *formal adjoint* is defined as

$$L^\dagger(f) = \sum_{|J| \leq d} (-1)^{|J|} D^J(a_J f) \ , \ \forall f \in K \ .$$

The formal adjoint possesses the following useful for the factorization theory properties:

$$(L^\dagger)^\dagger = L \ , \quad (L_1 \circ L_2)^\dagger = L_2^\dagger \circ L_1^\dagger \ , \quad \mathrm{Sym}_L = (-1)^{\mathrm{ord}(L)} \mathrm{Sym}_{L^\dagger} \ .$$

The property of having a factorization is invariant under the operation of taking the formal adjoint, while the property of having a factorization of certain factorization type is not invariant, and an operator L has a factorization of some factorization type $(S_1)(S_2)$ (where $\mathrm{Sym}_L = S_1 S_2$) if and only if $L^\dagger$ has that of factorization type $(S_2)(S_1)$.

Lemma 2. *The operation of taking the formal adjoint can be defined on the equivalent classes of LPDOs.*

Proof. Show that operation of taking the formal adjoint and the gauge transformations of LPDOs commute. For every $g \in K^*$, and $f = g^{-1}$ we have

$$(g^{-1} \circ L \circ g)^\dagger = g^\dagger \circ L^\dagger \circ (g^{-1})^\dagger = g \circ L^\dagger \circ g^{-1} = f^{-1} \circ L^\dagger \circ f.$$

Example 1 (LPDOs of order 2*).* For operators of the form

$$L = D_{xy} + aD_x + bD_y + c$$

there is a complete generating set of invariants that consists of first-order invariants: $h = c - a_x - ab$ and $k = c - b_y - ab$. For the formal adjoint

$$L^\dagger = D_{xy} - aD_x - bD_x + c - a_x - b_y$$

they are $h^\dagger = c - b_y - ab$ and $k^\dagger = c - a_x - ab$, and so $h_t = k$, $k_t = h$.

Theorem 6 (formal adjoint for equivalent classes). *Consider the equivalent classes of (4) given by the values of the invariants* I_1, I_2, I_3, I_4, I_5 *(5). Then the operation of taking of the formal adjoint is defined by the following formulae*

$$\left.\begin{array}{l}
I_1^\dagger = I_1 - 2q^2p_y - 2p^2q_x + 2p_xqp + 2q_yqp\ ,\\
I_2^\dagger = -I_2 - qp^2q_{xy} + q_yp^2q_x + q^2pp_{xy} - q^2p_xp_y\ ,\\
I_3^\dagger = -I_3 + \frac{1}{q^2}\Big(2pI_2 - (2p_yq + q_yp)I_1 + qpI_{1y} - 2p_yq_yq^2p +\\
\qquad\qquad 2q^3p_y^2 + q_{yy}q^2p^2 - q^3pp_{yy}\Big)\ ,\\
I_4^\dagger = -I_4 + \frac{1}{p^2}\Big(-2qI_2 - (p_xq + 2q_xp)I_1 + qpI_{1x} + 2p^3q_x^2 - 2p^2q_xqp_x\\
\qquad\qquad +p_{xx}q^2p^2 - qp^3q_{xx}\Big)\ ,\\
I_5^\dagger = I_5 + p_1I_1 + p_3I_3 + p_4I_4 + p_{12}I_{1y} + p_{11}I_{1x} + p^2I_{1xy} - qpI_{3x} - \frac{p^3}{q}I_{4y} + p_0\\
\qquad -pI_{2y} + \frac{p^2}{q}I_{2x} + (-2q^2p^3q_x + 4p_yq^4p - q^2pI_1 - 2q^3p^2p_x)/(q^4p)I_2\ ,
\end{array}\right\} \tag{13}$$

where $p_1 = (4q_xp_yp + p_xq_yp - 2q_{xy}p^2)/q + (4q_xq_yp^2)/q^2 + 3p_xp_y - p_{xy}p$, $p_3 = 2qp_x + pq_x$, $p_4 = (2q_yp^3 + p^2p_yq)/q^2$, $p_0 = p^3q_xq_{yy} - 2q^2p_xp_y^2 - qq_xp^2p_{yy} + q^2p_xpp_{yy} - qp^2q_{yy}p_x - 2p^2p_yq_yq_x + 2qq_xpp_y^2 + 2qp_ypq_yp_x$, $p_{11} = -(2p_ypq + q_yp^2)/q$, $p_{12} = -(p_xpq + 2q_xp^2)/q$.

Proof. Consider an operator L in the form (4) of some equivalent class and express the coefficients a_{11}, a_{10}, a_{01}, a_{00} of it in terms of the invariants (5) and a_{20}, a_{02}. Then compute the formal adjoint $L^\dagger$, and compute the invariants (5). The first invariant of $L^\dagger$ is already given in terms of the invariants of L and in the same form as in the statement of the theorem. The second invariant of $L^\dagger$ is

$$I_2^\dagger = qp^2a_{02y} - qp^2q_{xy} - a_{02}p^2q_y + q_yp^2q_x + q^2pp_{xy} - q^2a_{20x}p - q^2p_xp_y + q^2a_{20}p_x\ .$$

Employing the expression for the invariant I_2 we eliminate a_{20} and a_{02} from this expression and get $I_2^\dagger$ as it is in the statement of the theorem. Analogously, we obtain the forms for $I_3^\dagger$, $I_4^\dagger$ that are given in the statement of the theorem.

The fifth invariant $I_5^\dagger$ of $L^\dagger$ is a large expression containing a_{20} and a_{02}, and their second and first derivatives. The terms containing a_{02yy} are canceled if we add pI_{2y} to $I_5^\dagger$. Then the only term containing a_{20xx} is p^3qa_{20xx}, and we cancel it by subtraction of p^2I_{2x}/q. Then no second-order derivatives are left, and we notice that the ratio

$$C = (-2q^2p^3q_x + 4p_yq^4p - q^2pI_1 - 2q^3p^2p_x)/(q^4p)$$

of the coefficient for a_{20x} in the obtained expression to that in I_2 is equal to the ratio of the coefficient for a_{02y} in the obtained expression to that in I_2. Thus, subtracting CI_2, we cancel first-order derivatives, and have as the result the invariant expression

$$I_{55} = I_5 + p_1I_1 + p_3I_3 + p_4I_4 + p_{12}I_{1y} + p_{11}I_{1x} + p^2I_{1xy} - qpI_{3x} - \frac{p^3}{q}I_{4y} + p_0\ ,$$

where $p_1 = (4q_x p_y p + p_x q_y p - 2q_{xy}p^2)/q + (4q_x q_y p^2)/q^2 + 3p_x p_y - p_{xy}p$, $p_3 = 2qp_x + pq_x$, $p_4 = (2q_y p^3 + p^2 p_y q)/q^2$, $p_0 = p^3 q_x q_{yy} - 2q^2 p_x p_y^2 - qq_x p^2 p_{yy} + q^2 p_x p p_{yy} - qp^2 q_{yy} p_x - 2p^2 p_y q_y q_x + 2qq_x p p_y^2 + 2qp_y p q_y p_x$, $p_{11} = -(2p_y pq + q_y p^2)/q$, $p_{12} = -(p_x pq + 2q_x p^2)/q$ are differential-algebraic expressions of p and q. Thus,

$$I_5^\dagger = I_{55} - pI_{2y} + \frac{p^2}{q} I_{2x} + CI_2 \ .$$

Theorem 6 is the one that allows us to half the cases necessary to consider to describe existence of factorizations of different factorizations types. Below is an example on how to obtain invariant conditions of existence of a factorization of the certain type of factorizations $(XY)(pX + qY)$, if those are given (found in the section 4) for the "symmetric" factorization type $(pX + qY)(XY)$.

Corollary 3. *Consider an equivalent class of (4) given by the values of the invariants I_1, I_2, I_3, I_4, I_5 (5). Operators of the class have a factorization of factorization type $(XY)(pX + qY)$ if and only if*

$$\begin{cases} 0 = q_0 - qpI_2 + q^3 I_3 - p^3 I_4 \ , \\ 0 = p_0 + p_1 I_1 - 4pqq_x I_2 + p_3 I_3 + p_4 I_4 + \frac{q^3}{p} I_5 - q^4 I_{3x} \\ \qquad -(pq^2 q_y/2 + q^3 p_y) I_{1x} + p^3 q I_{4x} + pI_1 I_4 + (pq^3)/2 I_{1xy} \\ \qquad +pq^2 I_{2x} - 3pq^2 q_x/2 I_{1y} \ , \end{cases}$$

where q_0, p_0, p_1, p_3, p_4 are expressions of p, q and their derivations, more precisely, $q_0 = -2p_y q_y q^3 p + q_{yy} q^3 p^2 - q^4 p p_{yy} + q^3 p^2 p_{xy} - q^2 p^3 q_{xy} - p_{xx} q^2 p^3 + qp^4 q_{xx} + 2q^4 p_y^2 - 2p^4 q_x^2 - q^3 p p_x p_y + q_y q p^3 q_x + 2p^3 q_x q p_x$, $p_0 = 2pq^3 p_x q_x p_y - 2p^2 q^2 q_x q_y p_x + 2p^2 q^3 p_{xx} p_x + 8p^4 q q_x q_{xx} - 10p^4 q_x^3 - 5p^3 p_{xx} q^2 q_x - p^4 q^2 q_{xxx} + p^3 q^3 p_{xxx} - 5p^3 q^2 q_{xx} p_x - 4p^2 q^2 p_x^2 q_x - pq^4 p_{xx} p_y + 14p^3 q_x^2 q p_x + 2p^3 q q_x^2 q_y + p^2 q^3 p_{xx} q_y + p^2 q^3 q_{xx} p_y - p^3 q^2 q_y q_{xx} - 2p^2 q^2 q_x^2 p_y$, $p_1 = 3q^2 q_x p_y - 2pq q_x p_x - p^2 q q_{xx} - q^3 p_{xy} + 1/pq^3 p_x p_y + 2p^2 q_x^2 + pq^2 p_{xx} + 2pq q_x q_y - \frac{1}{2} pq^2 q_{xy}$, $p_3 = 2q^4 p_x/p + q^3 q_x$, $p_4 = 2p^2 p_x q + p^2 q q_y - 5p^3 q_x - p p_y q^2$.

Proof. Operators of the class have a factorization of factorization type $(XY)(pX + qY)$ if and only if their formal adjoints $L^\dagger$ have a factorization of factorization type $(-pX - qY)(XY)$, which by theorem 2 is true if and only if $-I_3^\dagger q^3 + I_4^\dagger p^3 + pq(-pI_{1tx} + qI_{1ty}) + pq(-q_y + p_x)I_1^\dagger + 2(-p_y q^2 + q_x p^2)I_1^\dagger - 3pqI_2^\dagger = 0$ and $I_{st} I_2^\dagger + I_{rt} - 2pq^2 I_{2tx} - q^3 I_{2ty} = 0$. Using the results of section 7, these conditions can be rewritten in terms of the five invariants (5) of L, and after simplifications the expressions given in the statement of the theorem can be obtained.

Consider the special case where p and q are constants. Then without loss of generality one can assume $p = q = 1$.

Corollary 4 (case of the symbol with constant coefficients). *An LPDO (4) with $p = q = 1$ has a factorization of factorization type $(XY)(X + Y)$ if and only if*

$$\begin{cases} I_3 = I_2 + I_4 \ , \\ 0 \ = I_5 + \frac{1}{2} I_{1xy} + I_4 I_1 \ . \end{cases}$$

8 Symbol of Constant Coefficients

In the criteria for the existence of factorizations of different factorization types, the coefficients p and q of the symbol, and their derivatives occur fairly often. Therefore, it is interesting to look at the structure of the formulae in the important particular case in which p and q are constants, and, therefore, there exists a normal form of the operator with the (principal) symbol $(X+Y)XY$. Thus, without loss of generality one can assume $p=q=1$, and then combining the results of the previous sections we obtain the necessary and sufficient conditions of the existence of factorizations for each of the 12 different types.

Theorem 7. *Consider equivalent classes of (4) possesing the property $p=q=1$, and given by the values of the invariants I_1, I_2, I_3, I_4, I_5 (5). Operators of the class have a factorization of factorization type*

$(S)(XY)$ *if and only if*

$$\left.\begin{array}{ll} I_3 - I_4 + I_{1x} - I_{1y} - 3I_2 & = 0 \ , \\ I_1 I_2 + I_r + 2I_{2x} + I_{2y} & = 0 \ , \end{array}\right\} \tag{14}$$

where $I_r = \frac{1}{2} I_{1xy} - I_{4y} + I_{4x} + I_5 + I_{1xx} + I_1 I_4 - I_1 I_{1x}$*;*

$(S)(X)(Y)$ *if and only if*

$$(14) \quad \& \quad I_2 - I_4 + I_{1x} = 0 \ ;$$

$(S)(Y)(X)$ *if and only if*

$$(14) \quad \& \quad -2I_2 - I_4 + I_{1x} = 0 \ ;$$

$(X)(SY)$ *if and only if*

$$I_4 = 0 \quad \& \quad I_{2x} + I_5 - I_{3x} + I_{1xy}/2 = 0 \ ; \tag{15}$$

$(X)(S)(Y)$ *if and only if*

$$(15). \quad \& \quad I_3 - I_{1y} - 2I_2 = 0 \ ;$$

$(X)(Y)(S)$ *if and only if*

$$(15). \quad \& \quad I_3 = I_2 \ ;$$

$(XY)(S)$ *if and only if*

$$I_4 = I_3 - I_2 \quad \& \quad I_{1xy}/2 + I_1 I_4 + I_5 = 0 \ .$$

$(YS)(X)$ *if and only if*

$$I_4 = I_{1x} - 2I_2 \quad \& \quad I_5 = I_1 I_2 \ .$$

$(XS)(Y)$ *if and only if*

$$I_3 - I_{1y} - 2I_2 = 0 \quad \& \quad I_5 = I_{2x} + I_{1xy}/2 \ ;$$

$(Y)(SX)$ *if and only if*

$$I_3 = 0 \quad \& \quad I_5 = (I_4 + I_2)_y + I_1 I_2 - I_{1xy}/2 \ ; \tag{16}$$

$(Y)(X)(S)$ *if and only if*

$$(16) \quad \& \quad I_4 = -I_2 \ ;$$

$(Y)(S)(X)$ *if and only if*

$$(16) \quad \& \quad I_4 - I_{1x} = -2I_2 \ ;$$

Theorem 8 (formal adjoint for equivalent classes). *Consider the equivalent classes of (4) possessing the properties $p = 1$ and $q = 1$ and which are given by the values of the invariants I_1, I_2, I_3, I_4, I_5 (5). Then the operation of taking of the formal adjoint is defined by the following formulae*

$$\left.\begin{array}{l} I_1^\dagger = I_1 \ , \\ I_2^\dagger = -I_2 \ , \\ I_3^\dagger = -I_3 + 2I_2 + I_{1y} \ , \\ I_4^\dagger = -I_4 - 2I_2 + I_{1x} \ , \\ I_5^\dagger = I_5 + I_{1xy} - I_{3x} - I_{4y} - I_{2y} + I_{2x} - I_1 I_2 \ . \end{array}\right\}$$

9 Conclusion

We obtained invariant necessary and sufficient conditions for the existence of factorizations extending given factorizations of the principal symbol of operators (any such factorization of the symbol corresponds to a factorization type). We defined the classical operation of taking the formal adjoint of an operator for the equivalent classes of the considered LPDOs. In particular, this result allows us to reduce the number of case considerations when finding an invariant definition of the property of the existence of a factorization. The existence criterium are found explicitly for the factorization types $(S)(XY)$, $(X)(YS)$, $(S)(X)(Y)$, $(X)(S)(Y)$, where $S = (pX + qY)$. Invariant conditions for the other eight possibilities of factorization types can be derived from these ones, and consideration of the most difficult case $(XY)(S)$ is provided as an example of such derivation.

For the future, it would be interesting to find such conditions in an algorithmic way for operators of general order. Another line of investigations might be the derivation of invariant conditions for generalized factorization in the sense of Tsarev [6].

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